the same cellars, three weeks later. The tadpoles were of a large size. I obtained the same result,—the full development of the frog in the absence of light; but in this experiment I had another object in view, that of observing the growth and obtaining the exact weight of the tadpoles before, during, and after their metamorphosis into a frog.

Dr. Edwards said that in his experiment "the tadpoles attained an extraordinary size, doubling or trebling their usual full weight;" but he unfortunately does not mention any particular weight, or how long the tadpoles were preserved alive; in fact there is nothing definite.

During my several years of experiments I did not observe any remarkable increase of weight or size as mentioned by Dr. Edwards, although my first experiment was from the ovum to the full development of the frog, and the two last when the tadpoles were approaching the period of their development.

In my first experiment on the ovum, I never obtained a tadpole more than 8 grains in weight in the absence of light; but I found in a pool in the neighbourhood a number of tadpoles, some between 11 and 15 grains in weight; seven of them weighed 15 grains each. Of these large tadpoles I took twenty for my experiment, weighing altogether 264 grains, and averaging about 13 grains each. After their transformation the frogs weighed 93 grains, averaging about $4\frac{1}{2}$ grains each,—those of 15 grains in the tadpole state only weighing 5 grains as frogs, having lost two-thirds of their weight during their metamorphosis.

Subsequent experiments have been in accordance with the above.

III. "Note on Internal Radiation." By George G. Stokes, M.A., Sec. R.S., Lucasian Professor of Mathematics in the University of Cambridge. Received December 28, 1861.

In the eleventh volume of the 'Proceedings of the Royal Society,' p. 193, is the abstract of a paper by Mr. Balfour Stewart, in which he deduces an expression for the internal radiation in any direction within a uniaxal crystal from an equation between the radiations incident upon and emerging from a unit of area of a plane surface, having an arbitrary direction, by which the crystal is supposed to be bounded. With reference to this determination he remarks (p. 196), "But the

internal radiation, if the law of exchanges be true, is clearly independent of the position of this surface, which is indeed merely employed as an expedient. This is equivalent to saying that the constants which define the position of the bounding surface must ultimately disappear from the expression for the internal radiation." This anticipation he shows is verified in the case of the expression deduced, according to his principles, for the internal radiation within a uniaxal crystal, on the assumption that the wave-surface* is the sphere and spheroid of Huygens.

In the case of an uncrystallized medium, the following is the equation obtained by Mr. Stewart in the first instance.

Let R, R′ be the external and internal radiations in directions OP, OP′, which are connected as being those of an incident and refracted ray, the medium being supposed to be bounded by a plane surface passing through O. Let OP describe an elementary conical circuit enclosing the solid angle $\delta\phi$, and let $\delta\phi'$ be the elementary solid angle enclosed by the circuit described by OP′. Let i, i' be the angles of incidence and refraction. Of a radiation proceeding along PO, let the fraction A be reflected and the rest transmitted; and of a radiation proceeding internally along P′O let the fraction A′ be reflected, and the rest transmitted. Then by equating the radiation incident externally on a unit of surface, in the directions of lines lying within the conical circuit described by OP, with the radiation proceeding in a contrary direction, and made up partly of a refracted and partly of an externally reflected radiation, we obtain

$$\mathrm{R}\cos i\,\delta\phi=(1-\mathrm{A}')\,\mathrm{R}'\cos i'\,\delta\phi'+\mathrm{AR}\cos i\,\delta\phi,$$

or

$$(1-\mathrm{A})\,\mathrm{R}\cos i\,\delta\phi=(1-\mathrm{A}')\,\mathrm{R}'\cos i'\,\delta\phi'. \quad . \quad . \quad . \quad . \quad (1)$$

In the case of a crystal there are *two* internal directions of refraction, OP_1, OP_2, corresponding to a given direction PO of incidence, the rays along OP_1, OP_2 being each polarized in a particular manner.

* To prevent possible misapprehension, it may be well to state that I use this term to denote the surface, whatever it may be, which is the locus of the points reached in a given time by a disturbance propagated in all directions from a given point; I do not use it as a name for the surface defined analytically by the equation

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(c^2+a^2)y^2-c^2(a^2+b^2)z^2+a^2b^2c^2=0.$$

As the term *wave-surface* in its physical signification is much wanted in optics, the surface defined by the above equation should, I think, be called *Fresnel's surface*, or *the wave-surface of Fresnel.*

Conversely, there are two directions, P_1O, P_2O, in which a ray may be incident internally so as to furnish a ray refracted along OP, and in each case no second refracted ray will be produced, provided the incident ray be polarized in the same manner as the refracted ray OP_1 or OP_2. In the case of a crystal, then, equation (1) must be replaced by

$$(1-A) R \cos i \, \delta\phi = (1-A_1) R_1 \cos i_1 \, \delta\phi_1 + (1-A_2) R_2 \cos i_2 \, \delta\phi_2. \quad (2)$$

In the most general case it does not appear in what manner, if at all, equation (2) would split into two equations, involving respectively R_1 and R_2. For if an incident ray PO were so polarized as to furnish only one refracted ray, say OP_1, a ray incident along P_1O and polarized in the same manner as OP_1 would furnish indeed only one refracted ray, in the direction OP, but that would be polarized differently from PO ; so that the two systems are mixed up together.

But if the plane of incidence be a principal plane, and if we may assume that such a plane is a plane of symmetry as regards the optical properties of the medium*, the system of rays polarized in and the system polarized perpendicularly to the plane of incidence will be quite independent of each other, and the equality between the radiation incident externally and that proceeding in the contrary direction, and made up partly of a refracted and partly of an externally reflected radiation, must hold good for each system separately. In this case, then, (2) will split into two equations, each of the form (1), R now standing for half the whole radiation, and R′, A′, &c. standing for R_1, A_1, &c., or R_2, A_2, &c., as the case may be. It need hardly be remarked that the value of A is different in the two cases, and that R′ has a value which is no longer, as in the case of an isotropic medium, alike in all

* According to Sir David Brewster (Report of the British Association for 1836, part ii. p. 13, and for 1842, part ii. p. 13), when light is incident on a plane surface of Iceland spar in a plane parallel to the axis, the plane of incidence, which is a principal plane, is not in general a plane of optical, any more than of crystalline symmetry as regards the phenomena of reflexion, although, as is well known, all planes passing through the axis are alike as regards internal propagation and the polarization of the refracted rays. Hence, strictly speaking, the statement as to the independence of the two systems of rays should be confined to the case in which the principal plane is also a plane of crystalline symmetry. As, however, the unsymmetrical phenomena were only brought out when the ordinary reflexion was weakened, almost annihilated, by the use of oil of cassia, we may conclude that under common circumstances they would be insensible.

directions. In determining according to Mr. Stewart's principles the internal radiation in any given direction within a uniaxal crystal, no limitation is introduced by the restriction of equation (1) to a principal plane, since we are at liberty to imagine the crystal bounded by a plane perpendicular to that containing the direction in question and the axis of the crystal.

Mr. Stewart further reduces equation (1) by remarking that in an isotropic medium, as we have reason to believe, $A'=A$, and that the same law probably holds good in a crystal also, so that the equal factors $1-A$, $1-A'$ may be struck out. Arago long ago showed experimentally that light is reflected in the same proportion externally and internally from a plate of glass bounded by parallel surfaces; and the formulæ which Fresnel has given to express, for the case of an isotropic medium, the intensity of reflected light, whether polarized in a plane parallel or perpendicular to the plane of incidence, are consistent with this law. In a paper published in the fourth volume of the Cambridge and Dublin Mathematical Journal (p. 1), I have given a very simple demonstration of Arago's law, based on the sole hypothesis that the forces acting depend only on the positions of the particles. This demonstration, I may here remark, applies without change to the case of a crystal whenever the plane of incidence is a plane of optical symmetry. It may be rendered still more general by supposing that the forces acting depend, not solely on the positions of the particles, but also on any differential coefficients of the coördinates which are of an even order with respect to the time,—a generalization which appears not unimportant, as it is applicable to that view of the mutual relation of the ether and ponderable matter, according to which the ether is compared to a fluid in which a number of solids are immersed, and which in moving as a whole is obliged to undergo local dislocations to make way for the solids.

On striking out the factors $1-A$ and $1-A'$, equation (1) is reduced to

$$\frac{R'}{R}=\frac{\cos i\,\delta\phi}{\cos i'\,\delta\phi'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

In the case of an isotropic medium, R and R' are alike in all directions, and therefore the ratio of $\cos i\,\delta\phi$ to $\cos i'\,\delta\phi'$ ought to be independent of i, as it is very easily proved to be. The same applies to a uniaxal crystal, so far as regards the ordinary ray. But as re-

gards the extraordinary, it is by no means obvious that the ratio should be expressible in the form indicated—as a quantity depending only on the direction OP′. Mr. Stewart has, however, proved that this is the case, independently of any restriction as to the plane of incidence being a principal plane, on the assumption that the wave-surface has the form assigned to it by Huygens.

It might seem at first sight that this verification was fairly adducible in confirmation of the truth of the whole theory, including the assumed form of the wave-surface. But a little consideration will show that such a view cannot be maintained. Huygens's construction links together the law of refraction and the form of the wave-surface, n a manner depending for its validity only on the most fundamental principles of the theory of undulations. The construction which Huygens applied to the ellipsoid is equally applicable to any other surface; it was a mere guess on his part that the extraordinary wave-surface in Iceland spar was an ellipsoid; and although the ellipsoidal form results from the imperfect dynamical theory of Fresnel, it is certain that rigorous dynamical theories lead to different forms of the wave-surface, according to the suppositions made as to the existing state of things. For every such possible form the ratio expressed by the right-hand member of equation (3) ought to come out in the form indicated by the left-hand member, and not to involve explicitly the direction of the refracting plane: and as it seemed evident that it could not be possible, merely by such general considerations as those adduced by Mr. Stewart, to distinguish between those surfaces which were and those which were not dynamically possible forms of the wave-surface, I was led to anticipate that the possibility of expressing the ratio in question under the form indicated was a general property of surfaces. The object of the present Note is to give a demonstration of the truth of this anticipation, and thereby remove from the verification the really irrelevant consideration of a particular form of wave-surface; but it was necessary in the first instance to supply some steps of Mr. Stewart's investigation which are omitted in the published abstract.

The proposition to be proved may be somewhat generalized, in a manner suggested by the consideration of internal reflexion within a crystal, or refraction out of one crystallized medium into another in optical contact with it. Thus generalized it stands as follows:—

Imagine any two surfaces whatsoever, and also a fixed point O;

imagine likewise a plane Π passing through O. Let two points P, P′, situated on the two surfaces respectively, and so related that the tangent planes at those points intersect each other in the plane Π, be called *corresponding points with respect to the plane* Π. Let P describe, on the surface on which it lies, an infinitesimal closed circuit, and P′ the "corresponding" circuit; let $\delta\phi$, $\delta\phi'$ be the solid angles subtended at O by these circuits respectively, and i, i' the inclinations of OP, OP′ to the normal to Π. Then shall the ratio of $\cos i\delta\phi$ to $\cos i'\delta\phi'$ be of the form [P] : [P′], where P depends only on the first surface and the position of P, and [P′] only on the second surface and the position of P′. Moreover, if either surface be a sphere having its centre at O, the corresponding quantity [P] or [P′] shall be constant.

It may be remarked that the two surfaces may be merely two sheets of the same surface, or even two different parts of the same sheet.

Instead of comparing the surfaces directly with each other, it will be sufficient to compare them both with the same third surface; for it is evident that if the points P, P′ correspond to the same point P_1, on the third surface, they will also correspond to each other. For the third surface it will be convenient to take a sphere described round O as centre with an arbitrary radius, which we may take for the unit of length. The letters P_1, i_1, ϕ_1 will be used with reference to the sphere.

Let the surface and sphere be referred to rectangular coordinates, O being the origin, and Π the plane of xy. Let x, y, z be the coordinates of P; ξ, η, ζ those of P_1. Then x, y, z will be connected by the equation of the surface, and ξ, η, ζ by the equation

$$\xi^2+\eta^2+\zeta^2=1.$$

According to the usual notation, let

$$\frac{dz}{dx}=p,\ \frac{dz}{dy}=q,\ \frac{d^2z}{dx^2}=r,\ \frac{d^2z}{dxdy}=s,\ \frac{d^2z}{dy^2}=t.$$

The equations of the tangent planes at P, P_1, X, Y, Z being the current coordinates, are

$$Z-z=p\,(X-x)+q\,(Y-y),$$
$$\xi X+\eta Y+\zeta Z=1;$$

and those of their traces on the plane of xy are

$$pX+qY=px+qy-z,$$
$$\xi X+\eta Y=1;$$

and in order that these may represent the same line, we must have

$$\xi = \frac{p}{px+qy-z}, \qquad \eta = \frac{q}{px+qy-z}. \quad . \quad . \quad (4)$$

To the element $dxdy$ of the projection on the plane of xy of a superficial element at P, belongs the superficial element $dS=\sqrt{1+p^2+q^2}\,dxdy$, and to this again belongs the elementary solid angle $\frac{\cos\nu dS}{\rho^2}$, where $\rho=\mathrm{OP}$, and ν is the angle between the normal at P and the radius vector. Hence the total solid angle within a small contour is $\frac{\cos\nu}{\rho^2}\sqrt{1+p^2+q^2}\iint dxdy$, the double integral being taken within the projection of that small contour. Also $\cos i=\frac{z}{\rho}$. Hence

$$\cos i\,\delta\phi = \frac{z\cos\nu}{\rho^3}\sqrt{1+p^2+q^2}\iint dxdy\,;$$

and applying this formula to the sphere by replacing $z\sqrt{1+p^2+q^2}$ by 1, ν by 0, and ρ by 1, we have

$$\cos i_1\delta\phi_1 = \iint d\xi d\eta,$$

the double integral being taken over the projection of the corresponding small area of the sphere.

Now by the well-known formula for the transformation of multiple integrals we have

$$\iint d\xi d\eta = \iint\left(\frac{d\xi}{dx}\frac{d\eta}{dy}-\frac{d\xi}{dy}\frac{d\eta}{dx}\right)dxdy\,;$$

and therefore

$$\frac{\cos i\delta\phi}{\cos i_1\delta\phi_1} = \frac{z\cos\nu\sqrt{1+p^2+q^2}}{\rho^3\left(\frac{d\xi}{dx}\frac{d\eta}{dy}-\frac{d\xi}{dy}\frac{d\eta}{dx}\right)}.$$

But the first of equations (4) gives

$$d\xi = \frac{(px+qy-z)dp-p(xdp+ydq)}{(px+qy-z)^2}$$

$$= \frac{\{(qy-z)r-pys\}\,dx+\{(qy-z)s-pyt\}\,dy}{(px+qy-z)^2}.$$

Similarly,

$$d\eta = \frac{\{(px-z)t-qxs\}\,dy+\{(px-z)s-qxr\}\,dx}{(px+qy-z)^2}.$$

Hence

$$\frac{d\xi}{dx}\frac{d\eta}{dy}-\frac{d\xi}{dy}\frac{d\eta}{dx}=\frac{\mathrm{V}}{(px+qy-z)^4},$$

where

$$\mathrm{V}=\{(qy-z)r-pys\}\{(px-z)t-qxs\}-\{(qy-z)s-pyt\}\{(px-z)s-qxr\}$$

$$=\{(qy-z)(px-z)-pqxy\}(rt-s^2)$$

$$=z(z-px-qy)(rt-s^2).$$

Hence

$$\frac{\cos i\delta\phi}{\cos i_1\delta\phi_1}=\frac{z\cos\nu\sqrt{1+p^2+q^2}}{\rho^3}\,\frac{(z-px-qy)^3}{z(rt-s^2)}.$$

But if ϖ be the perpendicular let fall from O on the tangent plane at P,

$$z-px-qy=\sqrt{1+p^2+q^2}.\varpi,$$

and therefore

$$\frac{\cos i\delta\phi}{\cos i_1\delta\phi_1}=\frac{\cos\nu.\varpi^3}{\rho^3}\,\frac{(1+p^2+q^2)^2}{rt-s^2}.$$

But $\varpi=\rho\cos\nu$. Also the quadratic determining the principal radii of curvature at P is

$$(rt-s^2)v^2+(\&c.)v+(1+p^2+q^2)^2=0;$$

and therefore if v_1, v_2 denote the principal radii of curvature,

$$v_1v_2=\frac{(1+p^2+q^2)^2}{rt-s^2}.$$

Hence

$$\frac{\cos i\delta\phi}{\cos i_1\delta\phi_1}=\cos^4\nu.v_1v_2, \quad \dots\dots\dots\dots \quad (5)$$

and

$$\frac{\cos i\delta\phi}{\cos i'\delta\phi'}=\frac{\cos i\delta\phi}{\cos i_1\delta\phi_1}.\frac{\cos i_1\delta\phi_1}{\cos i'\delta\phi'}=\frac{\cos^4\nu.v_1v_2}{\cos^4\nu'.v_1'v_2'}, \quad \dots \quad (6)$$

which proves the proposition enunciated.

In the particular case of an ellipsoid of revolution of which n is the axial and m the equatorial semi-axis, compared with a sphere of radius unity, both having their centres at O′, one of the principal radii of curvature is the normal of the elliptic section, which by the properties of the ellipse is equal to $\frac{m}{n}m'$, m' denoting the semi-conjugate diameter; and the other is the radius of curvature of the elliptic section, or $\frac{m'^3}{mn}$. Also ϖ is the perpendicular let fall from the centre

on the tangent line of the section. Hence from (5) or (6)

$$\frac{\cos i\delta\phi}{\cos i'\delta\phi'}=\frac{\varpi^4}{\rho^4}\cdot\frac{mm'}{n}\cdot\frac{m'^3}{mn}=\frac{\varpi^4 m'^4}{n^2\rho^4}=\frac{m^4n^2}{\rho^4},$$

since $\varpi m'=mn$. This agrees with Mr. Stewart's result(p. 197), since the R_e and $\frac{1}{2}R$ of Mr. Stewart are the same as the R' and R of equation (3).

IV. "On the Intensity of the Light reflected from or transmitted through a Pile of Plates." By George G. Stokes, M.A., Sec. R.S., Lucasian Professor of Mathematics in the University of Cambridge. Received January 1, 1862.

The frequent employment of a pile of plates in experiments relating to polarization suggests, as a mathematical problem of some interest, the determination of the mode in which the intensity of the reflected light, and the intensity and degree of polarization of the transmitted light, are related to the number of the plates, and, in case they be not perfectly transparent, to their defect of transparency.

The plates are supposed to be bounded by parallel surfaces, and to be placed parallel to one another. They will also be supposed to be formed of the same material, and to be of equal thickness, except in the case of perfect transparency, in which case the thickness does not come into account. The plates themselves and the interposed plates of air will be supposed, as is usually the case, to be sufficiently thick to prevent the occurrence of the colours of thin plates, so that we shall have to deal with intensities only.

On account of the different proportions in which light is reflected at a single surface according as the light is polarized in or perpendicularly to the plane of incidence, we must take account separately of light polarized in these two ways. Also, since the rate at which light is absorbed varies with its refrangibility, we must take account separately of the different constituents of white light. If, however, the plates be perfectly transparent, we may treat white light as a whole, neglecting as insignificant the chromatic variations of reflecting power. Let ρ be the fraction of the incident light reflected at the first surface of a plate. Then $1-\rho$ may be taken as the intensity cf